THE DISSIPATION FUNCTION IN THE THEORY OF PLASTIC MEDIA WITH STRAIN HARDENING

(O DISSIPATIVNOI FUNKTSII V TEORII UPROCHNIAIUSHCHIKHSIA PLASTICHESKIKH SRED)

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The equations of the theory of plastic media with strain hardening can be written in the form [1]

$$de_{ij} = de_{ij}^e + de_{ij}^p, \qquad de_{ij}^{\prime e} = \frac{\sigma_{ij}'}{2G}, \qquad e_{ii} = \frac{1-\nu}{E}\sigma_{ii}$$

$$de_{ij}^p = d\nu \frac{\partial f}{\partial \sigma_{ij}}, \qquad f(\sigma_{ij}, e_{ij}^p, \chi_i, k_i) = 0 \tag{1}$$

where σ_{ij} and e_{ij} are the components of the stress and strain tensors, respectively; e_{ij}^e and e_{ij}^p are the components of the elastic and plastic strains, respectively; the prime indicates components of deviators; $f = 0$ is the yield surface; χ_i are nonholonomic hardening parameters; and G, E, ν, and k_i are constants.

Let us establish the position of the yield surface in stress space. Obviously, the position is completely determined by the values of the parameters and constants e_{ij}^p, χ_i and k_i.

We now examine the rate of dissipation of mechanical energy

$$D = \sigma_{ij}\varepsilon_{ij}^p, \qquad \varepsilon_{ij}^p = \frac{de_{ij}^p}{dt} \tag{2}$$

In stress space the dissipation function is interpreted as the scalar product of the vector $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^p$. In accordance with the flow rule, the vector $\boldsymbol{\varepsilon}^p$, is directed along the normal to the yield surface. For a convex yield surface, the direction of the normal uniquely determines the point on the yield surface. The vector $\boldsymbol{\varepsilon}^p$ therefore, uniquely determines the corresponding vector $\boldsymbol{\sigma}$ and the scalar product

$$D = \boldsymbol{\sigma}\boldsymbol{\varepsilon}^p \tag{3}$$

For a yield surface which has singularitites (edges, vertices), it is obvious that different vectors $\boldsymbol{\varepsilon}^p$ can correspond to the same vector $\boldsymbol{\sigma}$; nevertheless, the scalar product is uniquely determined by specification of the vector $\boldsymbol{\varepsilon}^p$.

Analogously, if the yield surface has nonconcave parts, then one vector $\boldsymbol{\varepsilon}^p$ can correspond to different points of the yield surface. Nonetheless, the specification of the vector $\boldsymbol{\varepsilon}^p$ uniquely determines Expression (3).

Thus, the following relation must hold

$$D = D(\varepsilon_{ij}^p, e_{ij}^p, \chi_i, k_i) \tag{4}$$

In accordance with (2) and (4) we can obtain

$$\sigma_{ij}\, de_{ij}^p = D\,(de_{ij}^p / dt,\ e_{ij}^p,\ \chi_i,\ k_i)\, dt \tag{5}$$

The left-hand side of (5) does not depend on time. The right-hand side must be independent of the differential dt. Therefore, the dissipation function must be a homogeneous function of degree one in the components of the strain rate

$$D\,(\varepsilon_{ij}^p,\ e_{ij}^p,\ \chi_i,\ k_i) = \varepsilon_{ij}^p\, \partial D / \partial \varepsilon_{ij}^p \tag{6}$$

It follows from (5) and (6) that

$$(\partial D / \partial \varepsilon_{ij}^p - \sigma_{ij})\, \varepsilon_{ij}^p = 0 \tag{7}$$

For fixed parameters e_{ij}^p and χ_i we write the relation (2) in total differentials

$$\sigma_{ij} d\varepsilon_{ij}^p + \varepsilon_{ij}^p d\sigma_{ij} = (\partial D / \partial \varepsilon_{ij}^p)\, d\varepsilon_{ij}^p \tag{8}$$

From the flow rule for fixed parameters e_{ij}^p and χ_i it follows that

$$\varepsilon_{ij}^p d\sigma_{ij} = \mu \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} = 0, \quad \mu = \frac{d\nu}{dt} \tag{9}$$

Then from (8) we obtain

$$\sigma_{ij} = \partial D / \partial \varepsilon_j^p, \qquad D = D\,(\varepsilon_{ij}^p,\ e_{ij}^p,\ \chi_i,\ k_i) \tag{10}$$

Eqs. (10) are in agreement with (7).

We shall show that it is possible to construct a theory of plasticity which is based upon the definition of the dissipation function (2). We introduce the set of all possible strain rate components ε_{ij}^{p*}, for which

$$D\,(\varepsilon_{ij}^{p*},\ e_{ij}^p,\ \chi_i, k_i) \leqslant D\,(\varepsilon_{ij}^p,\ e_{ij}^p,\ \chi_i k_i) \tag{11}$$

We introduce a maximum principle analogous to the maximum principle of von Mises [2]

$$\sigma_{ji}\varepsilon_{ij}^p \geqslant \sigma_{ij}\varepsilon_{ij}^{p*} \tag{12}$$

The convexity (nonconcavity) of the level surfaces of the dissipation function and the flow rule

$$\sigma_{ij} = \lambda \frac{\partial D}{\partial \varepsilon_{ij}^p}, \quad \lambda = D \Big/ \left(\frac{\partial D}{\partial \varepsilon_{ij}^p}\, \varepsilon_{ij}^p\right) \tag{13}$$

follow from the inequality (12).

We shall assume that the function D is homogeneous of degree one in the components ε_{ij}^p, in this case $\lambda = 1$. The derivatives $\partial D / \partial \varepsilon_{ij}^p$ are homogeneous functions of degree zero in the components ε_{ij}^p. Therefore, the six relations can be regarded as functions of five variables, e.g. $\varepsilon_{ij}^p / \varepsilon_{11}^p$.

Assuming that the relations (13) can be solved for the $\varepsilon_{ij}^p / \varepsilon_{11}^p$, then as a result of elimination of the ε_{ij}^p we obtain some finite relations of the form (1) which do not contain strain rate components.

By differentiating the relation (2), we obtain Eq. (8). Using (10), we obtain from Eq. (8) that

$$\varepsilon_{ij}^p d\sigma_{ij} = 0 \tag{14}$$

Then by differentiating the relation obtained from (1) by fixing e_{ij}^p and χ_i, we find

$$\frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} = 0 \tag{15}$$

Eqs. (14) and (15) may be considered for corresponding values of σ_{ij} and ε_{ij}^p, from which it follows that there is a multiplier μ such that

$$\varepsilon_{ij}^p = \mu \frac{\partial f}{\partial \sigma_{ij}} \tag{16}$$

will hold.

The loading criterion is expressed in the form $D \geq 0$.

The correspondence between the yield function and the dissipation function is indicated in the Fig. 1.

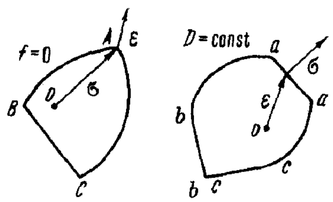

Fig. 1.

The convex portions ab and ac of the level surface of the dissipation function correspond to the convex portions AB and AC of the yield locus. The nonconvex portions aa, bb, and cc of the dissipation function correspond to the singularities A, B, and C of the yield function. Finally, the acute angle bc corresponds to the nonconvex portion BC.

If a singularity of the dissipation function is formed by the intersection of the smooth surfaces

$$D_m = D_m(\varepsilon_{ij}^p, e_{ij}^p, \chi_i, k_i) \qquad (m = 1, 2, \ldots, h)$$

then the following relation [3] holds:

$$\sigma_{ij} = \alpha_m (\partial D_m / \partial \varepsilon_{ij}^p), \qquad \alpha_m \geqslant 0$$
$$(\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1)$$

The definition of the dissipation function solves the problem of inversion of the relations between stress and strain in the theory of plastic media with strain hardening.

As an example, let us consider a variant of the theory of translational strain hardening

$$(\sigma_{ij} - ce_{ij}^p)(\sigma_{ij} - ce_{ij}^p) = \varphi(e_{ij}^p, \chi_i, k_i)$$

where c is a function of the invariants of the tensor e_{ij}^p and of the parameters χ_i.

According to the associated flow rule,

$$\varepsilon_{ij}^p = \mu(\sigma_{ij} - ce_{ij}^p) \tag{17}$$

Multiplying Eq. (17) by $\sigma_{ij} - ce_{ij}^p$ and summing on the scripts i and j, we obtain

$$(\sigma_{ij} - ce_{ij}^p)\varepsilon_{ij}^p = D - ce_{ij}^p \varepsilon_{ij}^p = \mu\varphi \tag{18}$$

Then multiplying (17) by ε_{ij}^p and summing on the scripts i and j, we obtain

$$\varepsilon_{ij}^p \varepsilon_{ij}^p = \mu(\sigma_{ij} - ce_{ij}^p)\varepsilon_{ij}^p = \mu(D - ce_{ij}^p \varepsilon_{ij}^p) \tag{19}$$

From Eqs. (18) and (19) we find the desired expression for the dissipation function

$$D = \sqrt{\varphi \varepsilon_{ij}^p \varepsilon_{ij}^p} + ce_{ij}^p \varepsilon_{ij}^p$$

According to (10), we obtain

$$\sigma_{ij} = \frac{\partial D}{\partial \varepsilon_{ij}^p} = \frac{\sqrt{\varphi}\, \varepsilon_{ij}^p}{\sqrt{\varepsilon_{ij}^p \varepsilon_{ij}^p}} + ce_{ij}^p \tag{20}$$

Taking into account that $\mu^2 = (\varepsilon_{ij}^p \varepsilon_{ij}^p)/\varphi$, we can obtain from (20) the original flow rule (17) and the yield surface. For $c = 0$ strain hardening occurs in which the yield surface experiences no translation.

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